UDC 62-50

DIFFERENTIAL-DIFFERENCE GAME OF ENCOUNTER WITH A FUNCTIONAL TARGET SET

PMM Vol. 37, №1, 1973, pp. 3-13
A. V. KRIAZHIMSKII and Iu. S. OSIPOV
(Sverdlovsk)
(Received August 21, 1972)

We establish sufficient conditions for the successful completion of a differential-difference game of encounter in the case when the target set is a subset of the space of initial states of the system. The paper is closely related to the investigations in [1 – 6].

1. Consider the system

$$x^{\cdot}(t) = A(t)\,x(t) + A_{\tau}(t)\,x(t-\tau) + B(t)\,u - C(t)\,v + w(t) \qquad (1.1)$$

Here x is the phase vector; vectors u and v are the controls of the first and second players, respectively, moreover,

$$u \in P(t), \quad v \in Q(t) \qquad (1.2)$$

where $P(t)$, $Q(t)$ are convex compacta continuous in t; the matrices $A(t)$, $A_{\tau}(t)$, $B(t)$, $C(t)$ are continuous; $w(t)$ is integrable on any interval of the t-axis; $\tau = \mathrm{const} > 0$. The segment $x_t(s) = x(t+s)$ of a trajectory of (1.1) (here and later s varies within the limits $-\tau \leqslant s \leqslant 0$) is called the state of system (1.1) at instant t.

Let H be the space of vector-valued functions $x(s)$ which are square summable in the quantity $\|x(s)\|$, with the norm

$$\|x(s)\|_{\tau} = \left(\|x(0)\|^2 + \int_{-\tau}^{0} \|x(s)\|^2\,ds\right)^{1/2}, \qquad \|x\|^2 = x_1^2 + \ldots + x_n^2$$

$\langle x, y\rangle$ is the scalar product in H. The game to be considered consists of the following [6]. We are given an initial instant t_0, an initial state $x^{\circ}(s) \in H$, a final instant $\vartheta \geqslant t_0$, and a certain set $M \subset H$ (the target). The first player, knowing at each instant $t \in [t_0, \vartheta]$ the state $x_t[\cdot] = x_t[s] = x[t+s]$ of the system, strives to choose his own control so that the final state $x_{\vartheta}[s]$ would lie in M. The second player chooses his control by any means and strives, to the contrary, to have $x_{\vartheta}[s] \notin M$. Let us make the problem statement more precise. We introduce some definitions [5, 6].

Definition 1.1. A function $u(t)$ $(v(t))$, summable on $[t_0, \vartheta]$ and satisfying the condition

$$u(t) \in P(t) \qquad (v(t) \in Q(t))$$

for almost all $t \in [t_0, \vartheta]$, is called the first (second) player's program control. The set of all program controls of the first (second) player is denoted $\{u\}$ $(\{v\})$.

Definition 1.2. 1°. A rule which associates the set

$$U(p) = U(t, x(s)) \subset P(t) \ (V(p) = V(t, x(s)) \subset Q(t))$$

with each pair $p = \{t, x(s)\}$, $t \in [t_0, \vartheta]$, $x(s) \in H$, named the position of the game, is called the first (second) player's strategy U (V) .

2°. A strategy U (V) of the first (second) player is said to be admissible if the set $U(t, x(s))$ ($V(t, x(s))$) defining this strategy is convex, closed, and upper semiconti-nuous relative to inclusion in t, $x(s)$ (in t, from the right).

3°. The first (second) player's trivial strategy U_T (V_T) is given by the sets $U(t, x(s)) = P(t)$ ($V(t, x(s)) = Q(t)$).

4°. The first (second) player's program strategy U_u (V_v) is given by the sets $U(t, x(s)) = \{u(t)\}$ ($V(t, x(s)) = \{v(t)\}$), where $u(t)$ ($v(t)$) is the first(second) player's program control.

Definition 1.3. 1°. Every function $x[t]$, absolutely continuous on $[t_0, \vartheta]$ and satisfying the condition

$$x[t_0 + s] = x^\circ(s) \tag{1.3}$$

and, for almost all $t \in [t_0, \vartheta]$, the equality

$$x^{\cdot}[t] = A(t)x[t] + A_\tau(t)x[t-\tau] + B(t)u[t] - C(t)v[t] + w(t)$$

where the summable functions $u[t]$ and $v[t]$ satisfy the conditions $u[t] \in U(t, x_t[s])$, $v[t] \in Q(t)$ for almost all $t \in [t_0, \vartheta]$, is called a motion $x[t, p_0, U, V_T]$ of system (1.1) from the position $p_0 = \{t_0, x^\circ(s)\}$, corresponding to the strategies U, V_T (U is admissible).

2°. An absolutely continuous function $x[t]$, satisfying condition (1.3) and, for almost all $t \in [t_0, \vartheta]$, the equality

$$x^{\cdot}[t] = A(t)x[t] + A_\tau(t)x[t-\tau] + B(t)u(t) - C(t)v(t) + w(t)$$

is called a motion $x[t, p_0, U_u, V_v]$ of system (1.1) from the position $p_0 = \{t_0, x^\circ(s)\}$, corresponding to the strategies U_u, V_v.

The system's motions defined in such a manner exist [7].

Problem 1. Given an initial position $p_0 = \{t_0, x^\circ(s)\}$, a final instant $\vartheta > t_0$, and a closed convex bounded set $M \subset H$ (the target). Construct the first player's admissible strategy U° such that all motions $x[t] = x[t, p_0, U^\circ, V_T]$ satisfy the condition $x_\vartheta[s] \in M$.

We also present the following definitions [6, 7].

Definition 1.4. The sets $W_t \subset H$, $t_0 \leqslant t \leqslant \vartheta$ are strongly u-stable if, whatever be $t_* \in [t_0, \vartheta)$, $t^* \in (t_*, \vartheta]$, $x(s) \in W_{t_*}$, $v(t) \in \{v\}$, there exists $u(t) \in \{u\}$ such that the motion $x[t] = x[t, \{t_*, x(s)\}, U_u, V_v]$ satisfies the condition $x_{t^*}[s] \in W_{t^*}$.

Definition 1.5. The set $W_{t_*}(\vartheta)$, $t_* \leqslant \vartheta$, of program absorption of target M by system (1.1) at the instant ϑ is the collection of all $x(s) \in H$ possessing the property: for any $v(t) \in \{v\}$ there exists $u(t) \in \{u\}$ such that the motion $x[t] = x[t, \{t_*, x(s)\}, U_u, V_v]$ satisfies the condition $x_\vartheta[s] \in M$.

In what follows it should be kept in mind that the concepts encountered below, which are not accompanied by explanations, are contained in [5, 6]. The following assertion stems from Lemma 4 of [6].

Theorem 1.1. Let the initial position $p_0 = \{t_0, x^\circ(s)\}$ be such that $x^\circ(s) \in W_{t_0}(\vartheta)$. If the sets $W_t(\vartheta)$, $t_0 \leqslant t \leqslant \vartheta$ are strongly u-stable, then the strategy U^e

extremal to them solves Problem 1.

On the basis of the theorem on the fixed point of a multivalued mapping, sufficient conditions were established in [6] for the strong u-stability of the program absorption sets of a finite-dimensional target in the general case of a nonlinear system with aftereffect. Such conditions were formulated in an analogous manner also for the problem of guidance onto a functional target (*). In the case of the linear system being considered we indicate the necessary and sufficient conditions (and the effective sufficient conditions ensuing from them) for the strong u-stability of the program absorption sets of a functional target. Let us state two auxiliary assertions analogous to the corresponding assertions in [5].

Lemma 1.1. $x(s) \in W_t(\vartheta)$ if and only if

$$\min_{\|h\|_\tau \leqslant 1}\{\rho(\vartheta, t, h) + \langle A_{t,\vartheta}x, h\rangle\} \geqslant 0 \tag{1.4}$$

Here

$$\rho(\vartheta, t, h) = r(\vartheta, t, h) - \min_{y \in M}\langle y, h\rangle$$

$$r(t^*, t_*, h) = r_1(t^*, t_*, h) - r_2(t^*, t_*, h) + r_3(t^*, t_*, h)$$

$$r_1(t^*, t_*, h) = \max_{u \in \{u\}}\left\langle h, \int_{t_*}^{t^*} F(t^* + s, \xi) B(\xi) u(\xi)\, d\xi \right\rangle$$

$$r_2(t^*, t_*, h) = \max_{v \in \{v\}}\left\langle h, \int_{t_*}^{t^*} F(t^* + s, \xi) C(\xi) v(\xi)\, d\xi \right\rangle$$

$$r_3(t^*, t_*, h) = \left\langle h, \int_{t^*}^{t^*} F(t^* + s, \xi) w(\xi)\, d\xi \right\rangle$$

$$A_{t_*, t^*}y(s) = F(t^* + s, t_*) y(0) +$$

$$\int_{-\tau}^{0} F(t^* + s, t_* + \tau + \eta) A_\tau(t_* + \tau + \eta) y(\eta)\, d\eta = f(s, y)$$

for $\delta = t^* - t_* \geqslant \tau$,

$$A_{t_*, t^*}y(s) = \begin{cases} f(s, y), & s \in [-\delta, 0] \\ y(s + \delta), & s \in [-\tau, -\delta) \end{cases}$$

for $\delta = t^* - t_* < \tau$, the matrix $F(\xi, \eta)$ satisfies the conditions: $F(\xi, \xi) = E$ is a unit matrix,

$$F(\xi, \eta) = 0 \quad \text{for} \quad \eta > \xi \tag{1.5}$$

$$\partial F(\xi, \eta)/\partial \xi = A(\xi) F(\xi, \eta) + A_\tau(\xi - \tau) F(\xi - \tau, \eta) \quad \text{for} \quad \eta < \xi.$$

Lemma 1.2. The sets $W_t(\vartheta)$, $t_0 \leqslant t \leqslant \vartheta$ are strongly u-stable if and only if

$$\inf_{h \in S}\{r(t^*, t_*, h) + \inf_{y \in W_{t_*}(\vartheta)}\langle A_{t_*, t^*}y, h\rangle - \inf_{y \in W_{t^*}(\vartheta)}\langle y, h\rangle\} \geqslant 0$$

for any $t_* \in [t_0, \vartheta)$, $t^* \in (t_*, \vartheta]$. Here S is the set of all $h \in H$, $\|h\|_\tau \leqslant 1$, on which the difference

$$\alpha(h) = \inf_{y \in W_{t_*}(\vartheta)}\langle A_{t_*, t^*}y, h\rangle - \inf_{y \in W_{t^*}(\vartheta)}\langle y, h\rangle$$

is defined (the values $\alpha(h) = \pm\infty$ are allowed).

---

*) This question was considered by Iu. S. Osipov: Problems in the Theory of Differential-Difference Games. Doctoral Dissertation, Sverdlovsk, 1971.

Let B_{t_*,t^*} be an operator adjoint to A_{t_*,t^*}, i.e. such that

$$\langle A_{t_*,t^*}x, h\rangle = \langle x, B_{t_*,t^*}h\rangle$$

for any h and x from H It is not difficult to establish that B_{t_*,t^*} has the form

$$B_{t_*,t^*}h(s) = T'(t^*, t_*, s)\,h(0) + \int_{-\tau}^{0} T'(t^* + \eta, t_*, s)\,h(\eta)\,d\eta = g(s, h)$$

for $\delta = t^* - t_* \geqslant \tau$,

$$B_{t_*,t^*}h(s) = \begin{cases} g(s, h), & s \in [-\tau, -\tau + \delta],\ s = 0 \\ h(s - \delta), & s \in (-\tau + \delta, 0) \end{cases}$$

for $\delta = t^* - t_* < \tau$. Here

$$T(t, \xi, s) = \begin{cases} F(t, \xi), & s = 0 \\ F(t, \xi + \tau + s)\,A_\tau(\xi + \tau + s), & s \in [-\tau, 0) \end{cases}$$

and the prime denotes transposition. From Lemma 1.2 and from the theorem on the separability of convex sets in space H there stem the following necessary and sufficient conditions for the strong u-stability of the program absorption sets $W_t(\vartheta)$.

Theorem 1.2. The sets $W_t(\vartheta)$, $t_0 \leqslant t \leqslant \vartheta$, are strongly u-stable if and only if

$$\inf_{\|h\|_\tau \leqslant 1}\{r(t^*, t_*, B_{t^*,\vartheta}h) + \inf_{y \in W_{t_*}(\vartheta)}\langle y, B_{t_*,\vartheta}h\rangle - \inf_{y \in W_{t^*}(\vartheta)}\langle y, B_{t_*,\vartheta}h\rangle\} \geqslant 0 \tag{1.6}$$

for any $t_* \in [t_0, \vartheta)$, $t^* \in (t_*, \vartheta]$, $t^* - t_* < \tau$.

2. The verification of condition (1.6) is difficult. Relying on Theorem 1.2, we indicate effective sufficient conditions for the strong u-stability of sets $W_t(\vartheta)$. By $W_t(\vartheta, \eta)$ we denote the program absorption set at instant ϑ of a closed η-neighborhood M^η of set M. By virtue of Lemma 1.1 and of the definition of the operator $B_{t,\vartheta}$,

$$W_t(\vartheta, \eta) = \{x(s) \mid \min_{\|h\|_\tau \leqslant 1}[\rho(\vartheta, t, h, \eta) + \langle x, B_{t,\vartheta}h\rangle] \geqslant 0\} \tag{2.1}$$

$$\rho(\vartheta, t, h, \eta) = \rho(\vartheta, t, h) + \eta\|h\|_\tau \tag{2.2}$$

Further, let the following conditions be fulfilled:

a) the function $\rho(\vartheta, t, h)$ is convex in h for all $t \in [t_0, \vartheta]$;

b) the sets $W_t(\vartheta, \eta)$ are not empty for all $\eta > 0$, $t \in [t_0, \vartheta]$.

We introduce the notation

$$A_t = A_{t,\vartheta}H, \qquad B_t = B_{t,\vartheta}H$$

if $h \in B_t$, then $K_t(h) = \{g \mid B_{t,\vartheta}g = h\}$. It is not difficult to establish that the subspace E_t of space H, orthogonal to the subspace $\bar{A}_t$ (th the closure of A_t), is the nucleus of the operator $B_{t,\vartheta}$. From this and from the fact that H is the direct sum of $\bar{A}_t$ and E_t, we obtain the following assertion.

Lemma 2.1. If $h \in B_t$, then there exists a unique element h_t from $K_t(h)$, belonging to $\bar{A}_t$, and

$$K_t(h) = h_t + E_t$$

We set

$$\rho_t(h, \eta) = \sup_{y \in W_t(\vartheta, \eta)}\langle y, h\rangle$$

Lemma 2.2. If $h \in B_t$, then

$$\rho_t(h, \eta) = \inf_{g \in K_t(h)}\rho(\vartheta, t, -g, \eta)$$

Proof. Let $h \in B_t$. In view of (2.1), for any $x \in W_t(\vartheta, \eta)$ and any $g \in K_t(h)$

$$\langle x, h\rangle \leqslant \rho(\vartheta, t, -g, \eta$$

whence

$$\rho_t(h, \eta) \leqslant \inf_{g \in K_t(h)} \rho(\vartheta, t, -g, \eta)$$

We show that for any $\varepsilon > 0$ there exists $x \in W_t(\vartheta, \eta)$ such that

$$\langle x, h\rangle > \inf_{g \in K_t(h)} \rho(\vartheta, t, -g, \eta) - \varepsilon \| h_t \|_\tau \quad (2.3)$$

Let $k \in \bar{A}_t$. We set $N(k) = k + E_t$. On $\bar{A}_t$ we define a functional

$$p(k) = \inf_{g \in N(k)} \rho(\vartheta, t, -g, \eta)$$

By Lemma 2.1, $N(l_t) = K_t(l)$, $l \in B_t$, therefore,

$$p(l_t) = \inf_{g \in K_t(l)} \rho(\vartheta, t, -g, \eta) \quad (2.4)$$

It is not difficult to establish that functional $p(k)$ is convex, positively homogeneous, and bounded.

Let us specify a subset L of space $\bar{A}_t$ in the following manner: y from $\bar{A}_t$ belongs to L if and only if $\langle y, k\rangle \leqslant p(k)$ for all $k \in \bar{A}_t$. It can be shown that $x \in W_t(\vartheta, \eta)$ if and only if $A_{t,\vartheta}\, x \in L$. Indeed, let $A_{t,\vartheta}\, x \in L$. Let g be an arbitrary element of H and let $B_{t,\vartheta}\, g = l$. Then, taking (2.4) into account, we obtain

$$\langle x, B_{t,\vartheta} g\rangle = \langle x, B_{t,\vartheta} l_t\rangle = \langle A_{t,\vartheta} x, l_t\rangle \leqslant p(l_t) \leqslant \rho(\vartheta, t, -g, \eta)$$

whence $x \in W_t(\vartheta, \eta)$. Conversely, let $x \in W_t(\vartheta, \eta)$. Let k be an arbitrary element of $\bar{A}_t$. For any $g \in N(k)$,

$$\langle x, B_{t,\vartheta}\, g\rangle + \rho(\vartheta, t, g, \eta) \geqslant 0$$

or, since $\langle A_{t,\vartheta}\, x, g\rangle = \langle A_{t,\vartheta}\, x, k\rangle$,

$$\langle A_{t,\vartheta}\, x, -k\rangle \leqslant \rho(\vartheta, t, g, \eta)$$

Because $g \in N(k)$ is arbitrary,

$$\langle A_{t,\vartheta} x, -k\rangle \leqslant \inf_{g \in N(k)} \rho(\vartheta, t, g, \eta) = p(-k)$$

hence, $A_{t,\vartheta}\, x \in L$. According to Theorem 2.2 in [7],

$$p(k) = \max_{y \in L} \langle y, k\rangle \quad (2.5)$$

Let $0 < \eta_1 < \eta$. Proceeding from (2.1) we can show that

$$A_{t,\vartheta} W_t(\vartheta, \eta_1) + \{y \in A_t \mid \| y \|_\tau \leqslant \eta - \eta_1\} \subset A_{t,\vartheta} W_t(\vartheta, \eta) \subset L$$

Then for an element $x_1 \in A_{t,\vartheta}\, W_t(\vartheta, \eta_1)$ we have, because L is closed in $\bar{A}_t$,

$$x_1 + S(\eta - \eta_1) \subset L$$

where $S(\eta - \eta_1)$ is a closed sphere in $\bar{A}_t$ of radius $\eta - \eta_1$ with center at 0. Let z be an element of L such that $\langle z, h_t\rangle = p(h_t)$. We set

$$C = \{y = \lambda x + (1 - \lambda) z \mid 0 \leqslant \lambda \leqslant 1, \ \ x \in x_1 + S(\eta - \eta_1)\}$$

Since L is convex, $C \subset L$. Let ε be an arbitrary positive number and let the element $y_1 = \lambda_1 x_1 + (1 - \lambda_1) z$ be such that $\| y_1 - z_1 \|_\tau < \varepsilon / 2$. Then, clearly, the $\lambda_1(\eta - \eta_1)$-neighborhood of element y_1 is contained in C. Since $C \subset L$, we conclude that in space

$\bar{A}_t$ some δ-neighborhood S_δ, $\delta < \varepsilon / 2$, of element y_1 lies in L. Since $y_1 \in \bar{A}_t$, we can find $y \in A_t = A_{t,\vartheta} H$ such that $y \in S_\delta$; therefore, $\| y - z \|_\tau < \varepsilon$. Since $S_\delta \subset L$, an element x such that $A_{t,\vartheta}\, x = y$, belongs to $W_t(\vartheta, \eta)$. Moreover, with due regard to (2.4), we have

$$\langle x, h\rangle = \langle x, B_{t,\vartheta} h_t\rangle = \langle A_{t,\vartheta} x, h_t\rangle \geqslant \langle z, h_t\rangle - \varepsilon \| h_t \|_\tau = p(h_t) - \varepsilon \| h_t \|_\tau =$$

$$\inf_{g \in K_t(h)} \rho(\vartheta, t, -g, \eta) - \varepsilon \| h_t \|_\tau$$

Relation (2.3) is proved. The lemma is proved.

Lemma 2.3. Let a function $z(\xi)$ be summable on $[t_*, t^*]$, $\delta = t^* - t_* < \tau$, $t^* < \vartheta$. Then

$$Z(s) = A_{t^*, \vartheta} \int_{t_*}^{t^*} F(t^* + s, \xi) z(\xi)\, d\xi = \int_{t_*}^{t^*} F(\vartheta + s, \xi) z(\xi)\, d\xi$$

Proof. At first let $\Delta = \vartheta - t^* \geqslant \tau$. Then, applying the Fubini theorem, we have

$$Z(s) = \beta(s) = \int_{t_*}^{t^*} \Phi(s, t^*, \xi) z(\xi)\, d\xi$$

$$\Phi(s, t, \xi) = F(\vartheta + s, t) F(t, \xi) + \int_{\xi - t^*}^{0} F(\vartheta + s, t + \tau + \eta) A_\tau(t + \tau + \eta) F(t + \eta, \xi)\, d\eta$$

From properties (1.5) of the matrix $F(\xi, \eta)$ we obtain

$$\partial \Phi(s, t, \xi) / \partial t = 0$$

for all $t, \xi, t > \xi$. Hence for all $\xi \in [t_*, t^*)$

$$\Phi(s, t^*, \xi) = \Phi(s, \xi, \xi) = F(\vartheta + s, \xi)$$

Consequently, the lemma's assertion is valid when $\Delta = \vartheta - t^* \geqslant \tau$.

Let $\Delta = \vartheta - t^* < \tau$. Then for $s \in [-\Delta, 0]$, as above, $Z(s) = \beta(s)$; for $s \in [-\tau, -\Delta)$

$$Z(s) = \int_{t_*}^{t^*} F(t^* + \Delta + s, \xi) z(\xi)\, d\xi = \beta(s)$$

The lemma is proved.

Theorem 2.1. Let conditions (a) and (b) be fulfilled. Then the set $W_t(\vartheta, \eta)$, $t_0 \leqslant t \leqslant \vartheta$ is strongly u-stable for any $\eta > 0$.

Proof. Let us show that condition (1.6) of Theorem 1.2 is fulfilled for any $t_* \in [t_0, \vartheta)$, $t^* \in (t_*, \vartheta]$, $t^* - t_* < \tau$. Let h_0 be an arbitrary element, $\| h_0 \|_\tau \leqslant 1$, $B_{t_*, \vartheta} h_0 = h_1$, $B_{t^*, \vartheta} h_0 = h_2$. Further, let g be an arbitrary element of $K_{t^*}(h_2)$. Using Lemma 2.3 and the expressions for $r_i(t^*, t_*, h)$ $(i = 1, 2, 3)$, we obtain

$$r_1(t^*, t_*, h_2) = \max_{u \in \{u\}} \left\langle g, \int_{t_*}^{t^*} F(\vartheta + s, \xi) B(\xi) u(\xi)\, d\xi \right\rangle = p_1(g)$$

$$r_2(t^*, t_*, h_2) = \max_{v \in \{v\}} \left\langle g, \int_{t_*}^{t^*} F(\vartheta + s, \xi) C(\xi) v(\xi)\, d\xi \right\rangle = p_2(g)$$

$$r_3(t^*, t_*, h_2) = \left\langle g, \int_{t_*}^{t^*} F(\vartheta + s, \xi) w(\xi)\, d\xi \right\rangle = p_3(g)$$

Consequently,

$$r(t^*, t_*, h_2) = p_1(g) - p_2(g) + p_3(g), \quad g \in K_{t^*}(h_2) \tag{2.6}$$

By Lemma 2.2,

$$\inf_{y \in W_{t^*}(\vartheta, \eta)} \langle y, h_2 \rangle = -\rho_t(-h_2, \eta) = -\inf_{k \in K_{t^*}(h_2)} \rho(\vartheta, t^*, k, \eta)$$

From this and from (2.6),

$$r(t^*, t_*, h_2) - \inf_{y \in W_{t^*}(\vartheta, \eta)} \langle y, h_2 \rangle = \inf_{g \in K_{t^*}(h_2)} [\rho(\vartheta, t^*, g, \eta) + r(t^*, t_*, h_2)] = \inf_{g \in K_{t^*}(h_2)} \{[r_1(\vartheta, t^*, g) + p_1(g)] - [r_2(\vartheta, t^*, g) + p_2(g)] + [r_3(\vartheta, t^*, g) + p_3(g)] + \eta \|h_2\|_k\} = \inf_{g \in K_{t^*}(h_2)} \rho(\vartheta, t_*, g, \eta) \tag{2.7}$$

Further,

$$\inf_{y \in W_{t_*}(\vartheta, \eta)} \langle y, h_1 \rangle = -\rho_{t_*}(-h_1, \eta) = -\inf_{g \in K_{t_*}(h_1)} \rho(\vartheta, t_*, g, \eta) \tag{2.8}$$

In view of (2.7), (2.8) the expression occurring under the inf sign in (1.6) equals, for $h = h_0$,

$$a(h_0) = \inf_{g \in K_{t^*}(h_2)} \rho(\vartheta, t_*, g, \eta) - \inf_{g \in K_{t_*}(h_1)} \rho(\vartheta, t_*, g, \eta) \tag{2.9}$$

Obviously,

$$K_t(B_{t,\vartheta} h_0) = h_0 + K_t(0) \tag{2.10}$$

From the definition of the operator A_{t_*, t^*} it follows that

$$B_{t^*, \vartheta} = B_{t_*, t^*} B_{t^*, \vartheta}$$

Therefore, if $B_{t^*, \vartheta} g = 0$, then $B_{t_*, \vartheta} g = 0$; consequently, $K_{t_*}(0) \supset K_{t^*}(0)$. Hence, from (2.9) and (2.10) and from the definition of the elements h_1, h_2 it follows that $a(h_0) \geqslant 0$. The theorem is proved, because h_0 is arbitrary.

Theorem 2.2. Let conditions (a) and (b) be fulfilled and let the set $W_{t_0}(\vartheta)$ not be empty. Then the sets $W_t(\vartheta)$, $t_0 \leqslant t \leqslant \vartheta$, are nonempty and strongly u-stable.

Proof. Let $t^* \in (t_0, \vartheta]$. We show that $W_{t^*}(\vartheta)$ is not empty. Let $x(s) \in W_{t_0}(\vartheta)$, $p = \{t_0, x(s)\}$, $v(t) \in \{v\}$. Let us prove that for some $u^*(t) \in \{u\}$ the motion $x^*[t] = x[t, p, U_{u^*}, V_v]$ satisfies the condition $x^*_{t^*}[s] \in W_{t^*}(\vartheta)$. Let $\eta_i \to 0$, $\eta_i > \eta_{i+1} > 0$, and $u_i(t) \in \{u\}$ be such that the motions $x^i[t] = x[t, p, U_{u_i}, V_v]$ satisfy the conditions $x_{t^*}{}^i[s] \in W_{t^*}(\vartheta, \eta_i)$. Such $u_i(t)$ exist since the sets $W_t(\vartheta, \eta_i)$, $t_0 \leqslant t \leqslant \vartheta$ are strongly u-stable according to Theorem 2.1 and $x(s) \in W_{t_0}(\vartheta) \subset W_{t_0}(\vartheta, \eta_i)$.

Since $\{u\}$ is weakly bicompact in $L_2[t_0, t^*]$, we can take it (by choosing a subsequence if necessary) that

$$u_i(t) \to u^*(t) \quad \text{weakly in } L_2[t_0, t^*] \tag{2.11}$$

By the Cauchy formula,

$$x^i[t] = A_{t_0, t} x(0) + \int_{t_0}^{t} F(t, \xi)[B(\xi) u_i(\xi) - C(\xi) v(\xi) + w(\xi)] d\xi \tag{2.12}$$

Proceeding from this expression we can show that the set of functions $\{x^i[t] \mid i = 1, 2, \ldots\}$ is uniformly bounded and equicontinuous on $[t_0, t^*]$, i.e. is compact in $C[t_0, t^*]$. Therefore, we can take it (by choosing a subsequence if necessary) that $x^i[t] \to y(t)$

in $C[t_0, t^*]$. On the other hand, from (2.11), (2.12) it follows that $x^i[t] \to x^*[t] = x[t, p, U_{u^*}, V_v]$ for any $t \in [t_0, t^*]$. Therefore, $x^*[t] = y(t)$, whence it follows that

$$x_{t^*}{}^i[s] \to x_{t^*}{}^*[s] \text{ in } H \tag{2.13}$$

Let us select an arbitrary element $h \in H$. Since $x_{t^*}{}^i \in W_{t^*}(\vartheta, \eta_i)$,

$$\langle x_{t^*}{}^i, h\rangle \leqslant \rho_{t^*}(h, \eta_i)$$

Since $\eta_i > \eta_{i+1}$, $W_{t^*}(\vartheta, \eta_i) \supset W_{t^*}(\vartheta, \eta_{i+1})$. Consequently, $\rho_{t^*}(h, \eta_i)$ decreases monotonically as i increases. Therefore, with due regard to (2.13), we have

$$\langle x_{t^*}{}^*, h\rangle = \lim_{i\to\infty} \langle x_{t^*}{}^i, h\rangle \leqslant \inf_i \rho_{t^*}(h, \eta_i) \tag{2.14}$$

Hence, $x_{t^*}{}^* \in \bigcap_{\eta>0} W_{t^*}(\vartheta, \eta)$. Indeed, if $x_{t^*}{}^* \notin \bigcap_{\eta>0} W_{t^*}(\vartheta, \eta)$, then a number i_0 exists such that $x_{t^*}{}^* \notin W_{t^*}(\vartheta, \eta_{i_0})$, therefore, $\langle x_{t^*}{}^*, h\rangle > \rho_{t^*}(h, \eta_{i_0})$ for some h, which contradicts (2.14) which is valid for all h. But, obviously, $\bigcap_{\eta>0} W_{t^*}(\vartheta, \eta) = W_{t^*}(\vartheta)$. We have proven that $W_{t^*}(\vartheta)$ is nonempty. The proof of the strong u-stability is a verbatim repetition of the proof carried out for the nonemptiness with the instant t_0 replaced everywhere by an arbitrary instant $t_* \in [t_0, t^*)$.

3. Let us ascertain the conditions necessary and sufficient for the fulfillment of assumption (b) (for the nonemptiness of all sets $W_t(\vartheta, \eta)$, $t_0 \leqslant t \leqslant \vartheta$, $\eta > 0$).

Lemma 3.1. $W_t(\vartheta, \eta)$ is nonempty for any $\eta > 0$ if and only if $\rho(\vartheta, t, h) \geqslant 0$ for all $h \in K_t(0) = \{h \mid B_{t,\vartheta} h = 0\}$.

Proof. Let $W_t(\vartheta, \eta)$ be nonempty for any $\eta > 0$; $y^{(\eta)} \in W_t(\vartheta, \eta)$. Then, in view of (2.1), for any h,

$$\langle y^{(\eta)}, B_{t,\vartheta} h\rangle + \rho(\vartheta, t, h, \eta) \geqslant 0$$

If $B_{t,\vartheta} h = 0$, then by (2.2)

$$\rho(\vartheta, t, h, \eta) = \rho(\vartheta, t, h) + \eta \| h \|_\tau \geqslant 0$$

whatever be $\eta > 0$; hence $\rho(\vartheta, t, h) \geqslant 0$.

Conversely, let $\rho(\vartheta, t, h) \geqslant 0$ for all $h \in K_t(0)$. On $\bar{A}_t$ we define a functional

$$q(k) = \inf_{g\in N(k)} \rho(\vartheta, t, -g), \qquad N(k) = k + E_t$$

It can be verified that under the assumptions adopted the functional $q(k)$ is convex, positively homogeneous, and bounded. Let $l \in \bar{A}_t$ be the support functional to $q(k)$ at the point $k = 0$ (such a functional exists [8]). We have

$$\min_{\|k\|_\tau\leqslant 1} [q(k) - \langle l, k\rangle] \geqslant 0 \tag{3.1}$$

Let $\eta > 0$ and let $p(k)$ be the functional on $\bar{A}_t$ introduced in the proof of Lemma 2.2,

$$p(k) = \inf_{g\in N(k)} \rho(\vartheta, t, -g, \eta)$$

Proceeding from (2.2) we can show that

$$p(k) \geqslant q(k) + \eta \|k\|_\tau \tag{3.2}$$

Let $l_1 \in A_t$, $\| l_1 - l \|_\tau < \eta$. Then from (3.1), (3.2), and the positive homogeneity of $p(k)$ it follows that $p(k) - \langle l_1, k\rangle \geqslant 0$, for all $k \in \bar{A}_t$, i.e. $l_1 \in L$ (Lemma 2.2). But then, as we have shown in the proof of Lemma 2.2, an element x such that $A_{t,\vartheta} x = l_1$ belongs to $W_t(\vartheta, \eta)$. The lemma is proved.

Lemma 3.2. If $\rho(\vartheta, t_0, h) \geqslant 0$ for all $h \in K_{t_0}(0)$, then, for any $t \in [t_0, \vartheta]$, $\rho(\vartheta, t, h) \geqslant 0$ for all $h \in K_t(0)$.

Proof. Suppose that the lemma's assumptions are fulfilled. Let us admit that for some $t^* > t_0$ there exists an element $h^* \in K_{t^*}(0)$ such that $\rho(\vartheta, t^*, h^*) < 0$. It is not difficult to verify that the equality

$$\left\langle h, \int_t^{\vartheta} F(\vartheta + s, \xi)\, z(\xi)\, d\xi \right\rangle = \int_t^{\vartheta} [B_{\xi,\vartheta} h(0)]' z(\xi)\, d\xi, \quad t \in [t_0, \vartheta],\ h \in H \tag{3.3}$$

is fulfilled for any summable function $z(\xi)$, $t_0 \leqslant \xi \leqslant \vartheta$. As was established in the proof of Theorem 2.1, $K_\xi(0) \supset K_{t^*}(0)$ for $\xi \leqslant t^*$. Hence, for $\xi \leqslant t^*$, $B_{\xi,\vartheta} h^* = 0$ in H, consequently, $B_{\xi,\vartheta} h^*(0) = 0$. Therefore, in view of (3.3) we have

$$\left\langle h^*, \int_{t_0}^{\vartheta} F(\vartheta + s, \xi)\, z(\xi)\, d\xi \right\rangle = \left\langle h^*, \int_{t^*}^{\vartheta} F(\vartheta + s, \xi)\, z(\xi)\, d\xi \right\rangle$$

From this it follows that $r_i(\vartheta, t_0, h^*) = r_i(\vartheta, t^*, h^*)$ $(i = 1, 2, 3)$; hence, $\rho(\vartheta, t_0, h^*) = \rho(\vartheta, t^*, h^*) < 0$, but this contradicts the assumption since $h^* \in K_{t^*}(0) \subset K_{t_0}(0)$. The lemma is proved.

The following assertion stems from Lemmas 3.1 and 3.2.

Theorem 3.1. Each of the following conditions is equivalent to condition (b):

c) $\rho(\vartheta, t_0, h) \geqslant 0$ for all $h \in K_{t_0}(0)$;

d) $W_{t_0}(\vartheta, \eta)$ is nonempty for all $\eta > 0$.

The following result ensues from Theorems 1.1, 2.2, 3.1.

Theorem 3.2. If the functional $\rho(\vartheta, t, h)$ is convex in h for all $t \in [t_0, \vartheta]$ (condition (a) is fulfilled) and $x^\circ(s) \in W_{t_0}(\vartheta)$, then the strategy U^e extremal to the sets $W_t(\vartheta)$, $t_0 \leqslant t \leqslant \vartheta$, solves Problem 1.

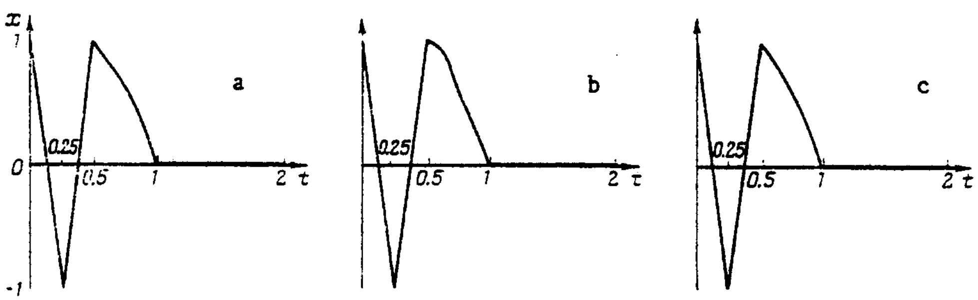

Fig. 1

Let us consider the following problem.

Problem 2. Given system (1.1), a closed convex bounded set $M \subset H$, an initial instant t_0, a final instant $\vartheta > t_0$, and a sequence of numbers $\varepsilon_i \to +0$. Find the sequences $\{x^i\}$ of elements of H and $\{U^i\}$ of admissible strategies of the first player such that the condition $x_\vartheta^i[s] \in M^{\varepsilon_i}$, where M^{ε_i} is the ε_i-neighborhood of set M is fulfilled for all motions $x^i[t] = x[t, \{t_0, x^i\}, U^i, V_T]$.

From Theorems 2.1, 3.1 follows

Theorem 3.3. If conditions (a) and (c) are fulfilled, then the following sequences $\{x^i\}$, $\{U^i\}$ solve Problem 2:

$$x^i \in W_{t_0}(\vartheta,\ \varepsilon_i)$$

U^i is the first player's strategy extremal to the sets $W_t\ (\vartheta,\ \varepsilon_i),\ t_0 \leqslant t \leqslant \vartheta$,

4. Problem 1 was simulated on an electronic computer for the system

$$x^{\cdot}(t) = x(t-1) + u - v \qquad (4.1)$$

where x, u, v are scalars, $|u| \leqslant 2,\ |v| \leqslant 1$, with $t_0 = 0,\ \vartheta = 2,\ M = \{0\} \subset H$. It is obvious that for system (4.1) the functional

$$\rho(\vartheta, t, h) = \max_{|z(\xi)| \leqslant 1} \left\langle h, \int_t^{\vartheta} F(\vartheta + s, \xi)\, z(\xi)\, d\xi \right\rangle$$

is convex in h. The function

$$x^{\circ}(s) = \begin{cases} -9, & -1 \leqslant s < 0.75 \\ 9, & -0.75 \leqslant s < -0.5 \\ -1.5, & -0.5 \leqslant s < 0 \\ 1, & s = 0 \end{cases}$$

was chosen as the initial state, lying in $W_{t_0}(\vartheta)$. Thus, by Theorem 3.1, the strategy U^e should solve Problem 1. Figure 1 a, b, c shows the trajectories which correspond to the strategy pairs $\{U^e, V_1\}$, $\{U^e, V_2\}$ and $\{U^e, V_3\}$, respectively. The strategies V_1, V_2 and V_3 are defined by the sets $V_1(t, x) = \{0\}$, $V_2(t, x) = \{v = -\operatorname{sgn} x(0)\}$ and $V_3(t, x) = \{v = -\operatorname{sgn} x(-1)/2\}$.

BIBLIOGRAPHY

1. Krasovskii, N. N., Game Problems on the Encounter of Motions. Moscow, "Nauka", 1970.
2. Pontriagin, L. S. and Mishchenko, E. F., Linear differential games. Dokl. Akad. Nauk SSSR, Vol.174, №1, 1967.
3. Pshenichnyi, B. N., On linear differential games. Kibernetika, №1, 1968.
4. Nikol'skii, M. S., Linear differential pursuit games in the presence of lags. Dokl. Akad. Nauk SSSR, Vol.197, №4, 1971.
5. Krasovskii, N. N. and Osipov, Iu. S., Linear differential-difference games. Dokl. Akad. Nauk SSSR, Vol.197, №4, 1971.
6. Osipov, Iu. S., Differential games of systems with aftereffect. Dokl. Akad. Nauk SSSR, Vol.196, №4, 1971.
7. Dem'ianov, V. F. and Rubinov, A. M., Approximate Methods for Solving Extremal problems. Izd. Leningradsk. Gos. Univ., 1968.
8. Pshenichnyi, B. N., Necessary Conditions for an Extremum. Moscow, "Nauka", 1969.

Translated by N.H.C.